

Integral representations of separable states ^{*}

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Abstract

We study a separability problem suggested by mathematical description of bipartite quantum systems. We consider hermitian 2-forms on the tensor product $H = K \otimes L$, where K, L are finite dimensional complex spaces. Such a form is called *separable* if it is a convex combination of hermitian tensor products $\sigma_p^* \odot \sigma_p$ of 1-forms σ_p on H that are product forms $\sigma_p = \varphi_p \otimes \psi_p$, where $\varphi_p \in K^*$, $\psi_p \in L^*$.

We introduce an integral representation of separable forms. We show that the integral of $D_{z^*}\Phi^* \odot D_z\Phi$ of any square integrable map $\Phi : \mathbb{C}^n \rightarrow \mathbb{C}^m$, with square integrable conjugate derivative $D_{z^*}\Phi$, is a separable form. Conversely, any separable form in the interior of the set of such forms can be represented in this way. This implies that any separable mixed state (and only such states) can be either explicitly represented in the integral form, or it may be arbitrarily well approximated by such states.

Keywords Bipartite systems, quantum states, separable states, entanglement, hermitian forms, separability problem

1 Introduction

Notions of separability and entanglement of states of a compound quantum system are of vital importance in quantum physics and quantum information theory. They emerged with the discovery of the EPR effect [1], however, the throughout analysis came much later [2, 3, 4, 5, 6]. Now the variety of theoretical problems where they play a central role is constantly growing (quantum cryptography [7], quantum teleportation [8], dense coding [9], ...) and many theoretical properties are confirmed in experiments [10, 11, 12].

Testing if a given state is separable or entangled (i.e. non-separable) seems one of the central questions in the theory of compound systems. Given a pure quantum state, it is easy to decide if it is separable or entangled. For mixed

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states this is not the case. At present there is no general and effective method to check if a given mixed state is separable or entangled and the problem seems hard [13]. The most effective necessary condition is the partial transpose test [4, 5], which is also sufficient in small dimensions [5].

In this work we present an indirect criterion for a mixed state of a bi-partite system to be separable. We introduce integral representations of separable states and prove that any state having the integral representation is separable. Vice versa, any state in the interior of the cone of separable states can be represented in the integral form.

To be more precise, denote $\mathcal{H} = \mathbb{C}^m \otimes (\mathbb{C}^n)^*$ and let $d\mu$ be the standard Lebesgue measure in $\mathbb{C}^n \simeq \mathbb{R}^{2n}$. With the use of identification $Hom(\mathbb{C}^n, \mathbb{C}^m) \simeq \mathbb{C}^m \otimes (\mathbb{C}^n)^*$ our main results (Theorems 3.1 and 3.2) can be stated as follows.

Theorem 1.1 (a) *For any square integrable map $\Phi : \mathbb{C}^n \rightarrow \mathbb{C}^m$ having the conjugate differential $D_{z^*}\Phi(z) \in \mathcal{H}$ square integrable, the density operator (non-normalized mixed state) $\mathcal{H} \rightarrow \mathcal{H}$ defined by*

$$\int_{\mathbb{C}^n} |D_{z^*}\Phi\rangle \langle D_{z^*}\Phi| d\mu(z) \quad (1.1)$$

is separable.

(b) *Any separable mixed state in the interior of the set of separable mixed states can be expressed in the above form.*

(c) *The above results also hold with \mathbb{C}^n replaced with the complex torus \mathbb{CT}^n .*

From mathematical view-point it is more convenient to state and prove our results in terms of hermitian 2-forms. In particular, using hermitian forms will not require the use of scalar product in the statement of our results.

Indeed, positive semi-definite hermitian 2-forms can be used to represent mixed states, instead of self-adjoint, positive semi-definite operators on a Hilbert space. If \mathcal{H} is an arbitrary Hilbert space, the obvious identification of these notions is given by the formula

$$\langle w|\rho_o v\rangle = \rho_f(w, v),$$

where ρ_o is a self-adjoint operator in \mathcal{H} and $\rho_f : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{C}$ is the corresponding hermitian form, with respect to the scalar product $\langle \cdot | \cdot \rangle$ in \mathcal{H} . In our case of $\mathcal{H} = \mathbb{C}^m \otimes (\mathbb{C}^n)^* = Hom(\mathbb{C}^n, \mathbb{C}^m)$ we use the scalar product

$$\langle A|B\rangle = \text{tr} A^\dagger B$$

and then the identification, in the standard basis, is simply given by

$$(\rho_o)_{ijkl} = (\rho_f)_{ijkl}.$$

2 Separable hermitian forms

Let H be a complex vector space. We will consider hermitian 2-forms on H , i.e., maps $\rho : H \times H \rightarrow \mathbb{C}$ which are \mathbb{C} -linear with respect to the second argument and anti-linear with respect to the first one. Given a linear function $f : H \rightarrow \mathbb{C}$, we denote by f^* its complex conjugate, $f^*(z) = (f(z))^*$, where in the latter case $*$ denotes complex conjugation in \mathbb{C} . Given linear functionals $\alpha, \beta : H \rightarrow \mathbb{C}$, we define their hermitian tensor product $\alpha^* \odot \beta : H \times H \rightarrow \mathbb{C}$ by

$$(\alpha^* \odot \beta)(z, w) = \frac{1}{2}(\alpha^*(z)\beta(w) + \beta^*(z)\alpha(w)),$$

which is a hermitian 2-form. In our considerations H will be the tensor product

$$H = K \otimes L$$

of complex vector spaces K, L of finite dimensions.

Definition 2.1 A hermitian 2-form $\rho : H \times H \rightarrow \mathbb{C}$ is called *separable* if it can be expressed as

$$\rho = \sum_{p=1}^P (\sigma^p)^* \odot \sigma^p,$$

where $P \geq 1$, $\sigma^p : H \rightarrow \mathbb{C}$ are linear functionals (elements of H^*) such that

$$\sigma^p = \varphi^p \otimes \psi^p,$$

with $\varphi^p \in K^*$ and $\psi^p \in L^*$, and $(\sigma^p)^*$ denotes complex conjugation of σ^p .

A form ρ is called *product* form if $\rho = \sigma^* \odot \sigma$, where $\sigma = \varphi \otimes \psi$, $\varphi \in K^*$ and $\psi \in L^*$. A positive semi-definite ρ is called *entangled* if it is not separable.

Note that separable hermitian 2-forms are positive semi-definite. The sets of separable (respectively, product) hermitian 2-forms on H will be denoted by \mathcal{C}_{sep} (resp. \mathcal{C}_{prod}). These are subsets of the real linear space \mathcal{C} of all hermitian 2-forms on H . Note that if the sum defining ρ is replaced by

$$\rho = \sum_{p=1}^P \lambda_p (\sigma^p)^* \odot \sigma^p,$$

with $\lambda_p \geq 0$ (equivalently, $\lambda_p \geq 0$ and $\sum_p \lambda_p = 1$) then we get an equivalent definition. Thus

$$\mathcal{C}_{sep} = \text{co} \mathcal{C}_{prod},$$

where $\text{co} A$ denotes the convex hull of A . Since $\dim \mathcal{C} = N^2$, where $N = \dim H$, it follows from the Carathéodory theorem that in the above sums we can always take $P \leq N^2$. The following fact is well known (as it is crucial in further considerations, we present its proof).

Proposition 2.2 *The set of separable hermitian 2-forms is a closed, convex cone with nonempty interior in the space of all hermitian 2-forms on H .*

Proof. Convexity comes from the above remarks. To prove closedness we consider the set S of separable hermitian 2-forms

$$\sum_{p=1}^{N^2} \lambda_p (\varphi^p \otimes \psi^p)^* \odot (\varphi^p \otimes \psi^p),$$

with $(\lambda_1, \dots, \lambda_{N^2})$ in the closed simplex defined by $\lambda_p \geq 0$ and $\sum_p \lambda_p = 1$, and φ^p, ψ^p in unit spheres in K^*, L^* (with respect to some fixed norms). The set S is a compact subset of the space of hermitian forms \mathcal{C} , as the image of a compact set under a suitable map. It does not contain the zero form, as all such forms are nontrivial, positive semi-definite. The cone \mathcal{C}_{sep} of all separable hermitian 2-forms is generated by S , thus \mathcal{C}_{sep} is closed.

To prove that \mathcal{C}_{sep} has nonempty interior in \mathcal{C} , it is enough to show that there is no nontrivial linear functional acting on \mathcal{C} which annihilates \mathcal{C}_{sep} . To begin with, let us fix hermitian products in K^* and L^* , and orthonormal basis $\epsilon_1, \dots, \epsilon_n$ in K^* and $\gamma_1, \dots, \gamma_m$ in L^* ($n = \dim K, m = \dim L$) with respect to these products. Define two sets of vectors in K^* and L^*

$$\begin{aligned} K_0 &= \{\epsilon_a + e_K \epsilon_b \mid a, b = 1, \dots, n, e_K = 1, i\} \subset K^*, \\ L_0 &= \{\gamma_c + e_L \gamma_d \mid c, d = 1, \dots, m, e_L = 1, i\} \subset L^*. \end{aligned}$$

Since $(\varphi \otimes \psi)^* \odot (\varphi \otimes \psi) \in \mathcal{C}_{sep}$, for $\varphi \in K_0, \psi \in L_0$, it is enough to show that

if an element $\theta \in \mathcal{C}^*$ of the dual space \mathcal{C}^* vanishes on every element $(\varphi \otimes \psi)^* \odot (\varphi \otimes \psi)$, with $\varphi \in K_0$ and $\psi \in L_0$, then $\theta \equiv 0$.

Denote $\theta_{ijkl} = \theta((\epsilon_i \otimes \gamma_j)^* \odot (\epsilon_k \otimes \gamma_l))$, so that $\theta_{ijkl} = \theta_{klij}^*$. We will successively show that $\theta_{ijkl} = 0$ for all $i, k = 1, \dots, n$ and $j, l = 1, \dots, m$. In every step we will be using the identities from the previous steps. Note first that if $a = b, c = d$ and $e_K = e_L = 1$ (i.e. $\epsilon_a + \epsilon_a \in K_0, \gamma_c + \gamma_c \in L_0$) then

$$0 = \theta((2\epsilon_a \otimes 2\gamma_c)^* \odot (2\epsilon_a \otimes 2\gamma_c)) = 16\theta_{acac},$$

and thus $\theta_{ijij} = 0$. Next, if we take $a \neq b, c = d$ and $e_L = 1$ then, using hermicity of θ , we obtain

$$0 = \theta\left((\epsilon_a + e_K \epsilon_b) \otimes 2\gamma_c\right)^* \odot ((\epsilon_a + e_K \epsilon_b) \otimes 2\gamma_c) = 8\text{Re}(e_K \theta_{acbc}),$$

therefore $\theta_{ijkj} = 0$, since real and imaginary parts of it vanish. Analogously we prove that $\theta_{ijil} = 0$. Finally, if we take $a \neq b, c \neq d$, we obtain

$$\begin{aligned} 0 &= \theta\left((\epsilon_a + e_K \epsilon_b) \otimes (\gamma_c + e_L \gamma_d)\right)^* \odot ((\epsilon_a + e_K \epsilon_b) \otimes (\gamma_c + e_L \gamma_d)) \\ &= 2(\text{Re}(e_K e_L \theta_{acbd}) + \text{Re}(e_K e_L^* \theta_{adbc})). \end{aligned}$$

Since four combinations of e_K and e_L give independent linear equations for real and imaginary parts of θ_{acbd} and θ_{adbc} , we conclude that $\theta_{ijkl} = 0$. \square

3 Integral representations

Let H_1, H_2 be vector spaces over \mathbb{C} and let

$$H = H_1 \otimes (H_2)^* = \text{Hom}(H_2, H_1).$$

The dual space $H^* = (H_1)^* \otimes H_2 \simeq H_2 \otimes (H_1)^*$ can be identified with the space of maps $\text{Hom}(H_1, H_2)$ and then the duality product is given by

$$\langle A, B \rangle = \text{tr}(AB) = \text{tr}(BA), \quad A \in H^*, \quad B \in H.$$

Given a \mathbb{C} -linear map $A : H_1 \rightarrow H_2$, we define a hermitian form $A \odot A$ on $H = \text{Hom}(H_2, H_1)$, which is the hermitian product of two copies of the linear functional $B \rightarrow \text{tr}(AB)$,

$$(A \odot A)(B, C) = \text{tr}(BA)^* \text{tr}(AC),$$

where $B, C \in \text{Hom}(H_2, H_1)$. This form is also given by the bilinear extension of $(A \odot A)(v \otimes w, \tilde{v} \otimes \tilde{w}) = (wAv)^* \tilde{w}A\tilde{v}$, for $w, \tilde{w} \in H_2^*$ and $v, \tilde{v} \in H_1$.

For a complex variable $z = x + iy$ and its complex adjoint $z^* = x - iy$ we use the usual notation $dz = dx + idy$, $dz^* = dx - idy$ for the complex 1-forms and $\partial_z = (\partial_x - i\partial_y)/2$ and $\partial_z^* = (\partial_x + i\partial_y)/2$ for the dual complex vector fields. Then $dz^* \wedge dz = 2idx \wedge dy$.

Let us assume that $H_1 = \mathbb{C}^n$ and $H_2 = \mathbb{C}^m$. We shall consider a map $\Phi = (\Phi_1, \dots, \Phi_m) : \mathbb{C}^n \rightarrow \mathbb{C}^m$ which is square integrable and, as a map from \mathbb{R}^{2n} to \mathbb{R}^{2m} , it has weak differential which is square integrable, too. We denote by $D_{z^*}\Phi(z) \in \mathbb{C}^m \otimes (\mathbb{C}^n)^*$ the conjugate differential of Φ at z which, by definition, is the complex linear map defined by the complex matrix

$$(D_{z^*}\Phi(z))_{ij} = \partial_{z_j}^* \Phi_i(z_1, \dots, z_n),$$

where $\partial_{z_j}^* = (\partial_{x_j} + i\partial_{y_j})/2$. The linear map $D_{z^*}\Phi(z) : \mathbb{C}^n \rightarrow \mathbb{C}^m$ can be considered as an element of the dual space H^* to the tensor product

$$H = \mathbb{C}^n \otimes (\mathbb{C}^m)^*.$$

Denote

$$dz^* \wedge dz = dz_1^* \wedge dz_1 \wedge \dots \wedge dz_n^* \wedge dz_n.$$

Theorem 3.1 (a) For an arbitrary square integrable map $\Phi : \mathbb{C}^n \rightarrow \mathbb{C}^m$ with square integrable conjugate differential $D_{z^*}\Phi(z)$, the hermitian 2-form $\rho_\Phi : H \times H \rightarrow \mathbb{C}$ defined by

$$\rho_\Phi = \frac{1}{(2i)^n} \int_{\mathbb{C}^n} (D_{z^*}\Phi(z))^* \odot D_{z^*}\Phi(z) dz^* \wedge dz \quad (3.1)$$

is separable.

(b) Any separable hermitian 2-form in the interior of the set of separable hermitian 2-forms can be expressed in the above form.

The same result holds with \mathbb{C}^n replaced by the complex torus. Recall that the n -dimensional complex torus is the quotient group

$$\mathbb{CT}^n = \mathbb{C}^n / \Lambda,$$

with topology and Lebesgue measure inherited from \mathbb{C}^n , where Λ is the lattice

$$\Lambda = \{(2\pi(a_1 + ib_1), \dots, 2\pi(a_n + ib_n)) \in \mathbb{C}^n \mid a_k, b_k \in \mathbb{Z}, k = 1, \dots, n\}$$

in \mathbb{C}^n . Given two points $z, \tilde{z} \in \mathbb{CT}^n$, there is a natural identification of the tangent spaces $T_z \mathbb{CT}^n$ and $T_{\tilde{z}} \mathbb{CT}^n$ via the standard parallel shift in \mathbb{C}^n . Therefore, as earlier, for any mapping $\Phi : \mathbb{CT}^n \rightarrow \mathbb{C}^m$ having the weak differential $D_{z^*} \Phi$ the linear map $D_{z^*} \Phi(z) : \mathbb{C}^n \rightarrow \mathbb{C}^m$ can be considered as element of the dual space $H^* = (\mathbb{C}^n)^* \otimes \mathbb{C}^m$.

Theorem 3.2 *Statements (a) and (b) of Theorem 3.1 hold if we replace \mathbb{C}^n with \mathbb{CT}^n , i.e., for square integrable maps $\Phi : \mathbb{CT}^n \rightarrow \mathbb{C}^m$, with square integrable conjugate differential $D_{z^*} \Phi$, and the integral is taken over \mathbb{CT}^n .*

Both theorems will be proved in the following two sections. We will also show (Theorem 7.1) that not all hermitian, positive semi-definite forms are integrally representable. Such forms lie in the boundary of the cone of all separable forms.

4 Separability of ρ_Φ

In proving statements (a) of both theorems we will use Fourier transform and the Hahn-Banach theorem.

The following elementary facts will be used in the proof. Given a function $f : \mathbb{C} \rightarrow \mathbb{C}$, we can write it as a complex valued function $\mathbb{R}^2 \rightarrow \mathbb{C}$ by identifying $f(x + iy) = f(x, y)$. Assuming that it is differentiable at $z = x + iy$, we have

$$\partial_z^* f(x + iy) = \frac{1}{2}(\partial_x f(x, y) + i\partial_y f(x, y)). \quad (4.1)$$

Consider the Fourier transform of $f : \mathbb{R}^2 \rightarrow \mathbb{C}$,

$$\widehat{f}(\xi, \zeta) = \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{-i(x\xi + y\zeta)} f(x, y) \, dx dy.$$

Then integration by parts gives

$$\widehat{\partial_x f}(\xi, \zeta) = i\xi \widehat{f}(\xi, \zeta), \quad \widehat{\partial_y f}(\xi, \zeta) = i\zeta \widehat{f}(\xi, \zeta).$$

Using (4.1) and taking $\kappa = \xi + i\zeta$, we aggregate this in the complex expression

$$\begin{aligned} \widehat{\partial_z^* f}(\kappa) &= \frac{1}{2} \left(\widehat{\partial_x f}(\xi, \zeta) + i\widehat{\partial_y f}(\xi, \zeta) \right) \\ &= \frac{1}{2} i \left(\xi \widehat{f}(\xi, \zeta) + i\zeta \widehat{f}(\xi, \zeta) \right) = \frac{1}{2} i \kappa \widehat{f}(\kappa). \end{aligned} \quad (4.2)$$

Let X be a finite dimensional vector space (or more generally, a Banach space). We shall need the following property, which follows from the Hahn-Banach theorem by a standard separation argument.

Proposition 4.1 *Let $S \subset X$ be a subset and $C \subset X$ be the smallest convex cone containing S . If C is closed and $x_0 \in X$ is an element satisfying*

$$\langle y, x_0 \rangle \geq 0, \quad \text{for any } y \in X^* \text{ such that } \langle y, x \rangle \geq 0 \text{ for all } x \in S,$$

then x_0 lies in C .

We will concentrate on a linear space \mathcal{C} over \mathbb{R} of all hermitian 2-forms $\rho : H \times H \rightarrow \mathbb{C}$ and its dual \mathcal{C}^* , with the duality product $\langle \cdot, \cdot \rangle : \mathcal{C}^* \times \mathcal{C} \rightarrow \mathbb{R}$. Take any linear coordinates in K and L . We have associated coordinates in \mathcal{C} and dual coordinates in \mathcal{C}^* . We can express $\theta = (\theta_{ijkl}) \in \mathcal{C}^*$ and $\rho = (\rho_{ijkl}) \in \mathcal{C}$, in these coordinates, where $\theta_{ijkl}^* = \theta_{klij}$ and $\rho_{ijkl}^* = \rho_{klij}$. Then

$$\langle \theta, \rho \rangle = \sum_{ijkl} \theta_{ijkl} \rho_{ijkl}.$$

Proof of Theorem 3.1 (a). Recall that \mathcal{C}_{prod} denotes the set of product hermitian 2-forms, and \mathcal{C}_{sep} the set of separable hermitian 2-forms. Since the convex hull of \mathcal{C}_{prod} is equal to \mathcal{C}_{sep} and it is a closed cone in \mathcal{C} then, by Proposition 4.1, $\rho_\Phi \in \mathcal{C}_{sep}$ if $\langle \theta, \rho_\Phi \rangle \geq 0$ for all $\theta \in \mathcal{C}^*$ such that

$$\langle \theta, \rho \rangle \geq 0, \quad \text{for any } \rho \in \mathcal{C}_{prod}.$$

Denote, for brevity, $\partial_j^* = \partial_{z_j}^*$. For $\theta \in \mathcal{C}^*$ in the conjugate cone to the cone of separable states and $\Phi = (\Phi_1, \dots, \Phi_m)$ we can write

$$\langle \theta, \rho_\Phi \rangle = \sum_{ijkl} \theta_{ijkl} \frac{1}{(2i)^n} \int_{\mathbb{C}^n} (\partial_j^* \Phi_i(z))^* \partial_l^* \Phi_k(z) dz^* \wedge dz.$$

Using Fourier transform and Parseval's equality $\int f^* g dz^* \wedge dz = \int \widehat{f^*} \widehat{g} d\kappa^* \wedge d\kappa$ we get

$$\int_{\mathbb{C}^n} (\partial_j^* \Phi_i(z))^* \partial_l^* \Phi_k(z) dz^* \wedge dz = \int_{\mathbb{C}^n} (\widehat{\partial_j^* \Phi_i}(\kappa))^* (\widehat{\partial_l^* \Phi_k}(\kappa)) d\kappa^* \wedge d\kappa.$$

Thus (4.2), and the fact that θ is positive on product states gives

$$\langle \theta, \rho_\Phi \rangle = \frac{1}{4(2i)^n} \int_{\mathbb{C}^n} \sum_{ijkl} \theta_{ijkl} (i\kappa_j \widehat{\Phi_i}(\kappa))^* (i\kappa_l \widehat{\Phi_k}(\kappa)) d\kappa^* \wedge d\kappa \geq 0,$$

which ends the proof. \square

Proof of Theorem 3.2 (a). The proof is analogous to the previous one. The only thing that one has to observe is the following. Given a square integrable

function $f : \mathbb{CT}^n \rightarrow \mathbb{C}$, its Fourier transform $\hat{f} : \mathbb{Z}^n \times \mathbb{Z}^n \rightarrow \mathbb{C}$ is given by the formula

$$\hat{f}(\alpha, \beta) = \frac{1}{(4\pi i)^n} \int_{\mathbb{CT}^n} e^{-i(\langle x, \alpha \rangle + \langle y, \beta \rangle)} f(z) dz^* \wedge dz,$$

where $z = x + iy \bmod 2\pi(\mathbb{Z} + i\mathbb{Z})$ are points in \mathbb{CT}^n , $\langle x, \alpha \rangle = \sum x_i \alpha_i$ and $\langle y, \beta \rangle = \sum y_i \beta_i$. Then integration by parts gives

$$\widehat{\partial_k^* f}(\alpha, \beta) = \frac{1}{2} i(\alpha_k + i\beta_k) \hat{\Phi}(\alpha, \beta),$$

and the Parseval's equality reads as

$$\frac{1}{(2i)^n} \int_{\mathbb{CT}^n} (f(z))^* g(z) dz^* \wedge dz = \sum_{(\alpha, \beta) \in \mathbb{Z}^{2n}} (\hat{f}(\alpha, \beta))^* (\hat{g}(\alpha, \beta)),$$

where $g : \mathbb{CT}^n \rightarrow \mathbb{C}$ is another square integrable function. \square

5 Construction of Φ in Theorem 3.1(b)

The construction of maps Φ which produce or approximate an arbitrary separable hermitian form is divided into three steps. First we will prove that any product hermitian 2-form can be arbitrarily closely approximated by the hermitian 2-forms ρ_Φ . Then we approximate any separable hermitian 2-form. Finally, we prove that any separable hermitian 2-form in the interior of all separable hermitian 2-forms can be expressed in the integral form (3.1).

Step 1. Approximation of product forms.

Consider the function $f_w : \mathbb{C}^n \rightarrow \mathbb{C}$ given by

$$f_w(z) = h_w(z) g_\alpha(z), \tag{5.1}$$

where $w \in \mathbb{C}^n$ and $\alpha > 0$ are fixed and

$$h_w(z) = e^{\langle z, w \rangle - \langle w, z \rangle} = e^{2i\text{Im}\langle z, w \rangle}, \quad g_\alpha(z) = (\pi\alpha)^{-n/2} e^{-\langle z, z \rangle / 2\alpha},$$

with $\langle z, w \rangle = \sum z_j^* w_j$. Clearly,

$$\partial_j^* f_w(z) = (w_j - \frac{z_j}{2\alpha}) f_w(z).$$

We have $|h_w(z)| = 1$ and, writing $z_j = x_j + iy_j$,

$$|f_w(z)| = \frac{1}{(\pi\alpha)^{n/2}} e^{-\sum (x_j^2 + y_j^2) / 2\alpha}.$$

This allows to prove the following lemma.

Lemma 5.1

$$\frac{1}{(2i)^n} \int_{\mathbb{C}^n} (\partial_j^* f_w(z))^* \partial_l^* f_w(z) dz^* \wedge dz = w_j^* w_l + \frac{1}{4\alpha} \delta_{jl}.$$

Proof. We have (all integrals are taken with respect to the measure $dz^* \wedge dz$)

$$\begin{aligned} \int_{\mathbb{C}^n} (\partial_j^* f_w(z))^* \partial_l^* f_w(z) &= \int_{\mathbb{C}^n} (w_j - z_j/2\alpha)^* (w_l - z_l/2\alpha) |f_w(z)|^2 \\ &= w_j^* w_l \int_{\mathbb{C}^n} |f_w(z)|^2 + \int_{\mathbb{C}^n} \frac{z_j^* z_l}{4\alpha^2} |f_w(z)|^2 \\ &\quad - \frac{w_j^*}{2\alpha} \int_{\mathbb{C}^n} z_l |f_w(z)|^2 - \frac{w_l}{2\alpha} \int_{\mathbb{C}^n} z_j^* |f_w(z)|^2 \\ &= (2i)^n (w_j^* w_l + \frac{1}{4\alpha} \delta_{jl}), \end{aligned}$$

where the first integral is computed using the Fubini theorem in the form

$$\begin{aligned} \int_{\mathbb{C}^n} |f_w|^2 &= \frac{1}{(\pi\alpha)^n} \prod_{s=1}^n \int_{\mathbb{C}^1} e^{-(x_s^2 + y_s^2)/\alpha} dz_s^* \wedge dz_s \\ &= \left(\frac{2i}{\pi\alpha}\right)^n \prod_{s=1}^n \int_{\mathbb{R}^1} e^{-x_s^2/\alpha} dx_s \int_{\mathbb{R}^1} e^{-y_s^2/\alpha} dy_s \end{aligned}$$

and the standard integral

$$\int_{\mathbb{R}} e^{-x^2/\alpha} dx = \sqrt{\pi\alpha}, \quad (5.2)$$

while in computing the second one with $j = l$ we use the above integral and

$$\int_{\mathbb{R}} x^2 e^{-x^2/\alpha} dx = \sqrt{\pi\alpha} \frac{\alpha}{2}. \quad (5.3)$$

The third and the fourth integrals are equal to zero as, after applying the Fubini theorem we obtain a factor of the form $\int_{\mathbb{C}} z_l e^{-|z_l|^2/\alpha} dz_l^* \wedge dz_l$, which is zero since the real and imaginary parts of the integrated function are antisymmetric with respect to corresponding (real or imaginary) axes. The same argument implies vanishing of the second integral, when $j \neq l$. \square

We shall approximate the hermitian 2-form $\sigma^* \odot \sigma$, where $\sigma = \varphi \otimes \psi$ and $\varphi \in \mathbb{C}^m$, $\psi \in (\mathbb{C}^n)^*$. To do that, let us take $\alpha > 0$ and consider the maps $\Phi : \mathbb{C}^n \rightarrow \mathbb{C}^m$ given by

$$\Phi(z) = \varphi \frac{1}{(\pi\alpha)^{n/2}} e^{\langle z, \psi \rangle - \langle \psi, z \rangle} e^{-\langle z, z \rangle/2\alpha} = \varphi f_\psi(z),$$

where f_ψ is the function defined in (5.1), with $w = \psi$, and we denote $\langle \psi, z \rangle = \sum \psi_i^* z_i$. By Lemma 5.1 we have

$$\begin{aligned} (\rho_\Phi)_{ijkl} &= \varphi_i^* \varphi_k \frac{1}{(2i)^n} \int_{\mathbb{C}^n} (\partial_j^* f_\psi(z))^* \partial_l^* f_\psi(z) dz^* \wedge dz \\ &= \varphi_i^* \varphi_k (\psi_j^* \psi_l + \frac{1}{4\alpha} \delta_{jl}). \end{aligned}$$

We see that, with α sufficiently large, the hermitian 2-form $\sigma^* \odot \sigma$ can be arbitrarily closely approximated by ρ_Ψ . \square

Step 2. Approximation of separable forms.

To prove that any separable state can be arbitrarily closely approximated by states in the integral form we use the result for product states. We start with a technical lemma.

Lemma 5.2 *For f_v and f_w of the form (5.1) we have*

$$\begin{aligned} \frac{1}{(2i)^n} \int_{\mathbb{C}^n} (\partial_j^* f_v(z))^* \partial_l^* f_w(z) dz^* \wedge dz &= \\ &= \frac{1}{4} \left((3v_j - w_j)^* (3w_l - v_l) + \frac{\delta_{jl}}{\alpha} \right) e^{-\alpha|v-w|^2}. \end{aligned}$$

Proof. Note that

$$\begin{aligned} \int_{\mathbb{C}^n} (\partial_j^* f_v(z))^* \partial_l^* f_w(z) &= v_j^* w_l \int_{\mathbb{C}^n} f_v^*(z) f_w(z) + \int_{\mathbb{C}^n} \frac{z_j^* z_l}{4\alpha} f_v^*(z) f_w(z) \\ &\quad - \frac{v_j^*}{2\alpha} \int_{\mathbb{C}^n} z_l f_v^*(z) f_w(z) - \frac{w_l}{2\alpha} \int_{\mathbb{C}^n} z_j^* f_v^*(z) f_w(z), \end{aligned}$$

where all integrals are taken with respect to $dz^* \wedge dz$. With $z_j = x_j + iy_j$ we have

$$\begin{aligned} f_v^*(z) f_w(z) &= h_v^*(z) h_w(z) g_\alpha^2(z) = e^{-2i\text{Im}\langle z, v \rangle} e^{2i\text{Im}\langle z, w \rangle} g_\alpha^2(z) \\ &= (\pi\alpha)^{-n} \prod_{s=1}^n e^{2ix_s \text{Im}(w_s - v_s)} e^{-2iy_s \text{Re}(w_s - v_s)} e^{-(x_s^2 + y_s^2)/\alpha}. \end{aligned}$$

Thus the proof of the lemma is a straightforward calculation analogous to the calculations in the proof of Lemma 5.1, with the use of three additional integrals

$$\begin{aligned} \int_{\mathbb{R}} e^{-x^2/\alpha} e^{i\gamma x} dx &= \sqrt{\pi\alpha} e^{-\frac{\alpha\gamma^2}{4}}, \\ \int_{\mathbb{R}} x e^{-x^2/\alpha} e^{i\gamma x} dx &= \sqrt{\pi\alpha} \frac{i\alpha\gamma}{2} e^{-\frac{\alpha\gamma^2}{4}}, \\ \int_{\mathbb{R}} x^2 e^{-x^2/\alpha} e^{i\gamma x} dx &= \sqrt{\pi\alpha} \frac{(2\alpha - \alpha^2\gamma^2)}{4} e^{-\frac{\alpha\gamma^2}{4}}. \end{aligned}$$

\square

Now consider a separable hermitian 2-form

$$\rho = \sum_{p=1}^P (\sigma^p)^* \odot \sigma^p,$$

where $\sigma^p = \varphi^p \otimes \psi^p$, $\varphi^p \in \mathbb{C}^m$, $\psi^p \in (\mathbb{C}^*)^n$ and we additionally require that $\psi^p \neq \psi^q$ for $p \neq q$, and $P \geq 1$. The set of such 2-forms is dense in the set of

all separable 2-forms. Thus, it is sufficient to approximate such 2-forms. To do that we consider the map $\Phi : \mathbb{C}^n \rightarrow \mathbb{C}^m$,

$$\Phi(z) = \sum_{p=1}^P \varphi^p f_{\psi^p} = \frac{1}{(\pi\alpha)^{n/2}} \sum_{p=1}^P \varphi^p e^{\langle z, \psi^p \rangle - \langle \psi^p, z \rangle} e^{-\langle z, z \rangle / 2\alpha}, \quad (5.4)$$

where we use formula (5.1) for f_{ψ^p} 's. By Lemma 5.2 we have

$$\begin{aligned} (\rho_\Phi)_{ijkl} &= \sum_{p,q} (\varphi_i^p)^* \varphi_k^q \frac{1}{(2i)^n} \int_{\mathbb{C}^n} (\partial_j^* f_{\psi^p}(z))^* \partial_l^* f_{\psi^q}(z) dz^* \wedge dz \\ &= \frac{1}{4} \sum_{p,q} (\varphi_i^p)^* \varphi_k^q \left((3\psi_j^p - \psi_j^q)^* (3\psi_l^q - \psi_l^p) + \frac{\delta_{jl}}{\alpha} \right) e^{-\alpha|\psi^p - \psi^q|^2}. \end{aligned}$$

Since $e^{-\alpha|\psi^p - \psi^q|^2} \rightarrow 0$, unless $p = q$, this expression converges to

$$\sum_{p=1}^P (\varphi_i^p)^* \varphi_k^p (\psi_j^p)^* \psi_l^p$$

when α tends to infinity. Thus we can approximate the form ρ with integrally representable forms ρ_Ψ . \square

From the above proof we can easily obtain the first part of the following proposition.

Proposition 5.3 *For any $P \in \mathbb{N}$, $\varphi_1, \dots, \varphi_P \in \mathbb{C}^m$, $\psi_1, \dots, \psi_P \in (\mathbb{C}^n)^*$ and $\alpha > 0$, the hermitian 2-form ρ with coefficients*

$$\rho_{ijkl} = \frac{1}{4} \sum_{p,q} (\varphi_i^p)^* \varphi_k^q \left((3\psi_j^p - \psi_j^q)^* (3\psi_l^q - \psi_l^p) + \frac{\delta_{jl}}{\alpha} \right) e^{-\alpha|\psi^p - \psi^q|^2} \quad (5.5)$$

is an integrally representable separable hermitian 2-form and $\rho = \rho_\Phi$ with Φ given in (5.4). Moreover, every hermitian 2-form in the interior of the cone of all separable 2-forms is of the form (5.5).

The second part of this proposition is equivalent to the statement of Theorem 3.1 (b). Therefore, to end the proof of the theorem we need to prove the proposition. Before we do that we make a few remarks about integrally representable hermitian 2-forms, which follow from (5.5).

Remark 5.4 Assume that $\varphi_p = \varphi$ for $p = 1, \dots, P$. Then (5.5) reduces to

$$\rho_{ijkl} = \frac{1}{4} \varphi_i^* \varphi_k \sum_{p,q} \left((3\psi_j^p - \psi_j^q)^* (3\psi_l^q - \psi_l^p) + \frac{\delta_{jl}}{\alpha} \right) e^{-\alpha|\psi^p - \psi^q|^2}$$

and ρ is of the form $(\varphi^* \odot \varphi) \otimes \tilde{\rho}$, where $\tilde{\rho}$ is a hermitian 2-form of rank n on \mathbb{C}^n (the rank of ρ is at least n , as we will see in Proposition 7.5).

Remark 5.5 If we take $\psi_p = \psi$, for $p = 1, \dots, P$, then the sum defining Φ in (5.4) has the same exponential function and it reduces to one summand (this is the simplest case considered in the first step of the proof). By the same reason we see that there is no loss of generality to assume in (5.4) that $\psi_p \neq \psi_q$, if $p \neq q$.

Step 3. Proof of the second part of Proposition 5.3.

Consider a separable hermitian 2-form ρ in the interior of $\mathcal{C}_{sep} \subset \mathcal{C}$, where \mathcal{C} is the real vector space of hermitian 2-forms. It follows from Proposition 2.2 that we can find $D = \dim \text{span} \mathcal{C}_{sep} = (nm)^2$ linearly independent separable hermitian 2-forms ρ^1, \dots, ρ^D , considered as vectors in the space \mathcal{C} of hermitian forms, such that $\rho = \sum_{d=1}^D \lambda_d^\rho \rho^d$ with $\lambda_1^\rho > 0, \dots, \lambda_D^\rho > 0$. Without loss of generality (changing slightly ρ^d , if necessary) we can assume that

$$\rho^d = \sum_{p=1}^{P_d} (\sigma^{(p,d)})^* \odot \sigma^{(p,d)},$$

where $P_d \geq 1$, $\sigma^{(p,d)} = \varphi^{(p,d)} \otimes \psi^{(p,d)}$, with $\varphi^{(p,d)} \in \mathbb{C}^m$, $\psi^{(p,d)} \in (\mathbb{C}^n)^*$ and $\psi^{(p,c)} \neq \psi^{(q,d)}$ for $(p,c) \neq (q,d)$. For each d consider a mapping $\Phi_d^\alpha : \mathbb{C}^n \rightarrow \mathbb{C}^m$ expressed by (5.4) with appropriate φ 's and ψ 's. Define the map $\Phi^\alpha : \mathbb{R}_+^D \times \mathbb{C}^n \rightarrow \mathbb{C}^m$,

$$\Phi^\alpha(\lambda_1, \dots, \lambda_D, z) = \sum_{d=1}^D \sqrt{\lambda_d} \Phi_d^\alpha(z).$$

Take $\alpha = 1/\beta^2$. We introduce the mapping $F : \mathbb{R}_+^D \times \mathbb{R} \rightarrow \mathcal{C} \simeq \mathbb{R}^D$ defined by

$$F(\lambda_1, \dots, \lambda_D, \beta) = \begin{cases} \frac{1}{(2i)^n} \int_{\mathbb{C}^n} (D_{z^*} \Phi^{\beta^{-2}}(\lambda, z))^* \odot D_{z^*} \Phi^{\beta^{-2}}(\lambda, z) dz^* \wedge dz, & \beta \neq 0 \\ \rho, & \beta = 0 \end{cases}.$$

From formula (5.5) obtained in Step 2 we know that

$$\begin{aligned} F(\lambda, \beta) &= \sum_{d=1}^D \sum_{p=1}^{P_d} \lambda_d (\sigma^{(p,d)})^* \odot \sigma^{(p,d)} + R(\lambda, \beta) \\ &= \sum_{d=1}^D \lambda_d \rho_d + R(\lambda, \beta), \end{aligned}$$

where $R : \mathbb{R}_+^D \times \mathbb{R} \rightarrow \mathcal{C}$ is a differentiable mapping of class C^∞ , given in

coordinates by

$$\begin{aligned}
R_{ijkl}(\lambda, \beta) &= \frac{1}{4} \sum_{d=1}^D \sum_{p=1}^{P_d} \lambda_d (\varphi_i^{(p,d)})^* \varphi_k^{(p,d)} \beta^2 \delta_{jl} + \\
&+ \frac{1}{4} \sum_{(p,c) \neq (q,d)} \sqrt{\lambda_c \lambda_d} (\varphi_i^{(p,c)})^* \varphi_k^{(q,d)} \\
&\times \left((3\psi_j^{(p,c)} - \psi_j^{(q,d)})^* (3\psi_l^{(q,d)} - \psi_l^{(p,c)}) + \beta^2 \delta_{jl} \right) e^{-\frac{1}{\beta^2} |\psi^{(p,c)} - \psi^{(q,d)}|^2}.
\end{aligned}$$

Since $F(\lambda^\rho, 0) = \rho$, we can complete the proof by using the implicit function theorem. We only have to prove that the rank of the differential of F with respect to λ is maximal at $(\lambda^\rho, 0)$. But this is trivial since $R(\cdot, 0) = 0$ ($R(\cdot, \beta)$ converges locally uniformly to zero function when β tends to zero) and ρ^1, \dots, ρ^D is a basis in $\mathcal{C} \simeq R^D$. \square

6 Construction of Φ in Theorem 3.2 (b)

Consider the family of mappings $\Phi : \mathbb{C}\mathbb{T}^n \rightarrow \mathbb{C}^m$ of the form

$$\Phi(z) = 2 \sum_{p=1}^P c_p^{-1} \varphi^p \chi_{a^p, b^p}(z), \quad (6.1)$$

where $z = (z_1, \dots, z_n)$ with $z_j = x_j + iy_j \pmod{2\pi(\mathbb{Z} + i\mathbb{Z})}$, $P \in \mathbb{N}$, and for all for $p = 1, \dots, P$ we take $\varphi^p \in \mathbb{C}^m$, $(a^p, b^p) \in \mathbb{Z}^n \times \mathbb{Z}^n$, $c_p \in \mathbb{N}$ and

$$\chi_{a^p, b^p}(z) = (2\pi)^{-n} e^{i(\langle x, a^p \rangle + \langle y, b^p \rangle)}.$$

Without losing generality we will assume that $(a^p, b^p) \neq (a^q, b^q)$ if $p \neq q$. Then, with $\psi^p = \frac{1}{c_p}(a^p + ib^p) \in \mathbb{Q}^n + i\mathbb{Q}^n$ and $\varphi^p \in \mathbb{C}^m$, we have

Lemma 6.1

$$\rho_\Phi = \sum_{p=1}^P (\varphi^p \otimes \psi^p)^* \odot (\varphi^p \otimes \psi^p).$$

Proof. Clearly,

$$\partial_j^* \Phi(z) = \sum_{p=1}^P i c_p^{-1} \varphi^p (a_j^p + i b_j^p) \chi_{a^p, b^p}(z),$$

where $\partial_j^* = \frac{1}{2}(\partial_{x_j} + i\partial_{y_j})$. Since

$$\frac{1}{(2i)^n} \int_{\mathbb{C}\mathbb{T}^n} (\chi_{a^p, b^p}(z))^* \chi_{a^q, b^q}(z) dz \wedge dz^* = \begin{cases} 1, & \text{if } p = q, \\ 0, & \text{otherwise,} \end{cases}$$

we find that

$$\begin{aligned} (\rho_\Phi)_{ijkl} &= \frac{1}{(2i)^n} \int_{\mathbb{CT}^n} (\partial_j^* \Phi_i(z))^* \partial_l^* \Phi_k(z) dz^* \wedge dz = \\ &= \sum_{p=1}^P c_p^{-2} (\varphi_i^p)^* (a_j^p + ib_j^p)^* \varphi_k^p (a_l^p + ib_l^p). \end{aligned}$$

Without coordinates one can write it as

$$\rho_\Phi = \sum_{p=1}^P (\varphi^p \otimes \psi^p)^* \odot (\varphi^p \otimes \psi^p), \quad (6.2)$$

where $\psi^p = \frac{1}{c^p} (a^p + ib^p)$. \square

Remark 6.2 Note that the set of forms

$$\sum_{p=1}^P (\varphi^p \otimes \psi^p)^* \odot (\varphi^p \otimes \psi^p),$$

with arbitrary $P \geq 1$, $\varphi^p \in \mathbb{C}^m$, and $\psi^p \in (\mathbb{Q} + i\mathbb{Q})^n$, is convex.

Proof of Theorem 3.2 (b). It follows from the lemma that every separable hermitian 2-form can be arbitrarily closely approximated by the forms ρ_Φ , since the set of ψ^p of the above form is dense in \mathbb{C}^n . This, together with the fact that the set of such forms is convex, implies that every hermitian 2-form in the interior of \mathcal{C}_{sep} is integrally representable. \square

Remark 6.3 Assume that we allow infinite summation in (6.1) with additional requirements that

$$\sum_{p=1}^{\infty} |c_p^{-1} \varphi^p|^2 < \infty, \quad \sum_{p=1}^{\infty} |c_p^{-1} \varphi^p (a_j^p + ib_j^p)|^2 < \infty, \quad (6.3)$$

for $j = 1, \dots, P$. Then it is obvious that (6.1) can be considered as a Fourier series, and every square integrable mapping from \mathbb{CT}^n to \mathbb{C}^m with square integrable differentials is of this form. Thus the set of separable hermitian 2-forms which are integrally representable on a torus is the closure of the set of the 2-forms ρ_Φ given by (6.2), in the topology given by the pseudonorms in (6.3). In particular, we have the following result.

Proposition 6.4 *A product form $\rho = \sigma^* \odot \sigma$ with $\sigma = \varphi \otimes \psi \neq 0$ is integrally representable on \mathbb{CT}^n iff the coordinates ψ_1, \dots, ψ_n of ψ are commensurable, i.e., there exist integers $a_1, \dots, a_n, b_1, \dots, b_n$ and a complex number $w \in \mathbb{C}$ such that $\psi_j = w(a_j + ib_j)$, for all j .*

Proof. The "if" part follows from Lemma 6.1 with $P = 1$. To prove the converse, assume that a product hermitian 2-form ρ is integrally representable. Then, by the above remark, it has a representation (6.2) with infinite sum and the additional requirements (6.3). Since ρ has rank one, there exist a product form $\varphi \otimes \psi \in \mathbb{C}^m \otimes (\mathbb{C}^n)^*$ and complex numbers w_1, w_2, \dots such that

$$\varphi^p \otimes \psi^p = w_p \varphi \otimes \psi, \quad p = 1, 2, \dots$$

Then, if $w_p \neq 0$, we can write $\varphi^p = u_p \varphi$, $\psi^p = v_p \psi$, with nonzero complex numbers u_p, v_p . Since $\psi^p = c_p^{-1}(a^p + ib^p)$, we get $\psi = w(a^p + ib^p)$, where $w = v_p^{-1} c_p^{-1}$. \square

7 Integral representation condition

As we have seen in Proposition 6.4, there are product forms that can be integrally represented on the torus. For the complex linear space this is not the case, as we will see in the following theorem.

Theorem 7.1 (a) *No product hermitian 2-form can be expressed in the integral form (3.1) over \mathbb{C}^n .*

(b) *Furthermore, if a hermitian 2-form ρ is representable in the integral form (3.1) over \mathbb{C}^n , then it satisfies the following condition.*

(IRC) *If there exist $v_0 \in (\mathbb{C}^m)^*$, $w_0 \in \mathbb{C}^n \setminus \{0\}$ such that $\rho(v_0 \otimes w_0, v_0 \otimes w_0) = 0$, then for all $w \in \mathbb{C}^n$ we have $\rho(v_0 \otimes w, v_0 \otimes w) = 0$.*

We will call (IRC) *the integral representation condition* (necessary for integral representability of ρ).

Proof. (a) It is enough to prove this statement for the hermitian 2-form $\rho_0 = \sigma^* \odot \sigma$ with $\sigma = \gamma_1 \otimes \epsilon_1$, where $\gamma_1, \dots, \gamma_m$ is the standard basis in \mathbb{C}^m and $\epsilon_1, \dots, \epsilon_n$ is the dual basis to the standard basis in \mathbb{C}^n . The general case reduces to this one by linear changes of coordinates in \mathbb{C}^m and \mathbb{C}^n .

We will show that there is no map Φ such that $\rho_\Phi = (\gamma_1 \otimes \epsilon_1)^* \odot (\gamma_1 \otimes \epsilon_1)$. Assume that there is such a map. Then from the definition of ρ_Φ we have

$$\begin{aligned} \delta_{i1} \delta_{k1} &= (\rho_\Phi)_{i1k1} = \frac{1}{(2i)^n} \int_{\mathbb{C}^n} (\partial_1^* \Phi_i(z))^* \partial_1^* \Phi_k(z) dz^* \wedge dz, \\ 0 &= (\rho_\Phi)_{ijkl} = \frac{1}{(2i)^n} \int_{\mathbb{C}^n} (\partial_j^* \Phi_i(z))^* \partial_l^* \Phi_k(z) dz^* \wedge dz \quad \forall j \neq 1 \text{ or } l \neq 1 \end{aligned}$$

For $i = k$ and $j = l$, the above equations imply that for any $i = 1, \dots, m$

$$\begin{aligned} \|\partial_1^* \Phi_i\|_{L^2} &= \delta_{1i}, \\ \|\partial_j^* \Phi_i\|_{L^2} &= 0, \quad j = 2, \dots, n, \end{aligned}$$

where $\|\cdot\|_{L^2}$ denotes the standard L^2 norm. From these expressions we deduce that for $i \neq 1$

$$\partial_j^* \Phi_i = 0 \quad \text{a.e.,} \quad j = 1, \dots, n.$$

Now from a remark to Theorem 4.6.10 in [14] it follows that, for $i \neq 1$, the maps $\Phi_i : \mathbb{C}^n \rightarrow \mathbb{C}$ are holomorphic on \mathbb{C}^n , and since we assume that they are square integrable, we have

$$\Phi_i = 0 \quad \forall i \neq 1.$$

For $i = 1$ we have

$$\partial_j^* \Phi_1 = 0 \quad \text{a.e.} \quad \forall j \neq 1.$$

Again from the remark mentioned above we obtain that for all $z_1 \in \mathbb{C}$ the function

$$\Phi_{z_1} := \Phi_1(z_1, \cdot, \dots, \cdot) : \mathbb{C}^{n-1} \rightarrow \mathbb{C}$$

is holomorphic on \mathbb{C}^{n-1} . Trying to find any nontrivial square integrable function Φ_1 to our problem, take any square integrable function $g : \mathbb{C}^n \rightarrow \mathbb{C}$ which fulfils compatibility conditions $\partial_j^* g = 0$ for all $j \neq 1$. Then by Theorem 4.6.11 in [14], there exists locally square integrable solution to the equations

$$\partial_j^* \Phi_1 = \delta_{1j} g, \quad j = 1, \dots, n. \quad (7.1)$$

If, additionally, for some g the solution Φ_1 is square integrable then, by the Fubini theorem, Φ_{z_1} must be square integrable function for almost all z_1 . This means that

$$\Phi_{z_1} = 0, \quad \text{for almost all } z_1,$$

as the null function is the only holomorphic square integrable function on \mathbb{C}^{n-1} . In consequence,

$$\|\Phi_1\|_{L^2} = 0$$

which, together with $\Phi_i = 0$ for $i \neq 1$, contradicts the inequality $\rho_\Phi \neq 0$.

(b) Just like in the proof of (a), we can assume that $v_0 = \tilde{\gamma}_1 \in (\mathbb{C}^m)^*$ and $w_0 = \tilde{\epsilon}_1 \in \mathbb{C}^n$, where $\tilde{\gamma}_1, \dots, \tilde{\gamma}_m$ is the dual basis to the standard basis in \mathbb{C}^m and $\tilde{\epsilon}_1, \dots, \tilde{\epsilon}_n$ is the standard basis in \mathbb{C}^n (the case of $v = 0$ is trivial). By the assumption, there exists a square integrable map $\Phi : \mathbb{C}^n \rightarrow \mathbb{C}^m$, such that $\rho = \rho_\Phi$, in particular,

$$0 = \rho(v_0 \otimes w_0, v_0 \otimes w_0) = \frac{1}{(2i)^n} \int_{\mathbb{C}^n} |\partial_1^* \Phi_1(z)|^2 dz^* \wedge dz.$$

Thus $\partial_1^* \Phi_1 = 0$ and, by arguments as above, we deduce that $\Phi_1(z_1, z_2, \dots, z_n)$ is a holomorphic function of z_1 , for almost all $(z_2, \dots, z_n) \in \mathbb{C}^{n-1}$. A holomorphic function in $L^2(\mathbb{C})$ must be identically zero, thus $\Phi_1 = 0$. Therefore, for all $w \in \mathbb{C}^n$ we have

$$\begin{aligned} \rho(v_0 \otimes w, v_0 \otimes w) &= \sum_{ijkl} \delta_{1i} w_j^* \delta_{1k} w_l \frac{1}{(2i)^n} \int_{\mathbb{C}^n} (\partial_j^* \Phi_i(z))^* \partial_l^* \Phi_k(z) dz^* \wedge dz \\ &= \sum_{jl} w_j^* w_l \frac{1}{(2i)^n} \int_{\mathbb{C}^n} (\partial_j^* \Phi_1(z))^* \partial_l^* \Phi_1(z) dz^* \wedge dz = 0, \end{aligned}$$

which is our assertion. \square

We will now analyze the integral representation condition. Consider a tensor product $H = K \otimes L$, where K and L are vector spaces over \mathbb{C} of finite dimensions m and n , respectively. Let $\rho : H \times H \rightarrow \mathbb{C}$ be a nonzero separable hermitian 2-form. We denote by $\bar{\rho} : H \rightarrow \mathbb{R}$ the quadratic form associated with ρ , i.e.

$$\bar{\rho}(v) = \rho(v, v).$$

Define two sets $\ker_K \rho \subset K$ and $\ker_L \rho \subset L$ by

$$\begin{aligned} \ker_K \rho &:= \{v \in K \mid \forall w \in L \quad \bar{\rho}(v \otimes w) = 0\}, \\ \ker_L \rho &:= \{w \in L \mid \forall v \in K \quad \bar{\rho}(v \otimes w) = 0\}. \end{aligned}$$

Let

$$\rho = \sum_{p=1}^P (\sigma^p)^* \odot \sigma^p,$$

where $P \geq 1$ and $\sigma^p \in H^*$ are linear functionals $\sigma^p = \varphi^p \otimes \psi^p$, with $\varphi^p \in K^*$ and $\psi^p \in L^*$. Then $\ker_K \rho$ is the intersection of the kernels of φ^p 's, and $\ker_L \rho$ is the intersection of the kernels of ψ^p 's i.e.

$$\ker_K \rho = \bigcap_p \ker \varphi^p \subset K, \quad \ker_L \rho = \bigcap_p \ker \psi^p \subset L.$$

Indeed, if v belongs to the intersection of the kernels of all φ^p 's, then it is obvious that $v \in \ker_K \rho$. Vice versa, if there exists p_0 such that $\varphi^{p_0}(v) \neq 0$, then $\bar{\rho}(v \otimes w) > 0$ for all $w \notin \ker \psi^{p_0}$, as all terms $\sigma^{p*} \odot \sigma^p$ defining $\bar{\rho}$ are nonnegative quadratic forms. The proof of the second equality is analogous. From these equalities it follows that $\ker_K \rho \subset K$ and $\ker_L \rho \subset L$ are linear subspaces of nonzero codimensions (since $\rho \neq 0$).

Let $v = v_1 + v_k \in K$, where $v_k \in \ker_K \rho$. Then

$$\bar{\rho}(v \otimes w) = \bar{\rho}(v_1 \otimes w).$$

Thus, for any vector $[v] \in K / \ker_K \rho$ the quadratic form $\bar{\rho}_{[v]} : L \rightarrow \mathbb{R}$ given by

$$\bar{\rho}_{[v]}(w) = \bar{\rho}(v \otimes w)$$

is well defined.

Proposition 7.2 *For any separable hermitian 2-form ρ the following conditions are equivalent.*

1. (IRC) *If there exist $v_0 \in K$ and $w_0 \in L \setminus \{0\}$ such that $\bar{\rho}(v_0 \otimes w_0) = 0$, then for all $w \in L$ we have $\bar{\rho}(v_0 \otimes w) = 0$.*
2. *If $v \in K$, $w \in L \setminus \{0\}$ and $\bar{\rho}(v \otimes w) = 0$ then $v \in \ker_K \rho$.*
3. *For all $0 \neq [v] \in K / \ker_K \rho$ the quadratic form $\bar{\rho}_{[v]}$ is strictly positive definite.*

Proof. (1 \Rightarrow 2) This implication is obvious.

(2 \Rightarrow 3) Assume that there exists $0 \neq [v] \in K/\ker_K \rho$ such that $\bar{\rho}_{[v]}$ is not strictly positive definite. Then there exist $w \in L \setminus \{0\}$ such that $\bar{\rho}_{[v]}(w) = 0$. From the definition of $\bar{\rho}_{[v]}$ and condition 2 we have that $v \in \ker_K \rho$ and therefore $[v] = 0$.

(3 \Rightarrow 1) Assume that there exists $v \in K, w \in L \setminus \{0\}$ such that $\bar{\rho}(v \otimes w) = 0$. Then condition 3 implies that $[v] = 0 \in K/\ker_K \rho$ i.e. $v \in \ker_K \rho$ and therefore for all $w \in L$ we have $\bar{\rho}(v \otimes w) = 0$. \square

Corollary 7.3 *If the integral representation condition holds for a nonzero separable hermitian 2-form ρ then $\ker_L \rho = 0$. In particular, if ρ is integrally representable on \mathbb{C}^n , then $\ker_L \rho = 0$.*

Proof. Assume that $\ker_L \rho \neq \{0\}$, and take any $0 \neq v \in K \setminus \ker_K \rho$ and $w \in \ker_L \rho \setminus \{0\}$. Then $\bar{\rho}(v \otimes w) = 0$, which contradicts condition 2 in Proposition 7.2. The second statement follows from Theorem 7.1. \square

Remark 7.4 Note that the negation of (IRC) gives an almost sufficient condition for entanglement of hermitian 2-forms. Namely, if a positive definite hermitian 2-form does not satisfy (IRC), then it belongs to the boundary of the cone of separable 2-forms, or it is entangled, by Theorem 3.1 (b) and Theorem 7.1.

Proposition 7.5 *If a nonzero hermitian 2-form ρ is integrally representable then it has rank at least $n = \dim L$.*

Proof. As we have mentioned above, since $\rho \neq 0$, codimension of $\ker_K \rho$ in K is nonzero. Thus $K/\ker_K \rho$ is nonempty. Therefore, using Theorem 7.1(b) and condition 3 in Proposition 7.2, we deduce that if ρ is integrally representable then there exists $[v] \in K/\ker_K \rho$ such that the quadratic form $\bar{\rho}_{[v]}$ on L is strictly positive definite i.e. has rank n . Thus there exists n -dimensional subspace in $K \otimes L$ such that the quadratic form $\bar{\rho}$ is strictly positive when restricted to this subspace. Thus its rank is at least n . \square

One could expect that the rank of the integrally representable hermitian 2-form ρ always equals $\text{codim}(\ker_K \rho) \cdot \dim(L)$. An example presented below contradicts this assertion. The example is a special case of the following proposition, which gives an interesting class of integrally representable forms.

Proposition 7.6 *Let $f : \mathbb{C}^n \rightarrow \mathbb{C}$ be a twice differentiable function (in the real sense), such that its first and second conjugate derivatives are square integrable. Denote by*

$$\Phi = D^* f = (\partial_1^* f, \dots, \partial_n^* f) : \mathbb{C}^n \rightarrow \mathbb{C}^n$$

the conjugate gradient of f . Then the hermitian 2-form ρ_Φ given by (3.1) is separable on $\mathbb{C}^n \otimes (\mathbb{C}^n)^$ and its kernel contains all antisymmetric tensors.*

Proof. From Theorem 3.1 the hermitian 2-form ρ_Φ is separable. So we need to check that $\bar{\rho}_\Phi(c) = 0$ for every antisymmetric tensor c . But this is obvious since the second conjugate derivative is a symmetric operator and therefore

$$\begin{aligned}\bar{\rho}_\Phi(c) &= \sum_{ijkl} \frac{1}{(2i)^n} \int_{\mathbb{C}^n} (c_{ij} \partial_i^* \partial_j^* f(z))^* c_{kl} \partial_k^* \partial_l^* f(z) dz^* \wedge dz \\ &= \frac{1}{(2i)^n} \int_{\mathbb{C}^n} \left| \sum_{ij} c_{ij} \partial_i^* \partial_j^* f(z) \right|^2 dz^* \wedge dz = 0, \quad \text{if } c_{ij} = -c_{ji}.\end{aligned}$$

□

Example 7.7 Consider a function $f : \mathbb{C}^n \rightarrow \mathbb{C}$ given by

$$f(z) = f_\psi(z) = \left(\frac{2}{\sqrt{\pi\alpha}} \right)^n e^{-\frac{1}{2\alpha^2}(4|z|^2 - 4\alpha^2 \langle \psi, z^* \rangle + \alpha^4 |w|^2)},$$

where $\alpha > 0$ and $\psi \in (\mathbb{C}^n)^*$. Denote by $\Phi(z) = D^* f(z)$ the conjugate differential of this function and consider the mapping $\Phi : \mathbb{C}^n \rightarrow \mathbb{C}^n$. Thus Φ determines a separable hermitian 2-form ρ_ψ on $\mathbb{C}^n \otimes (\mathbb{C}^n)^*$ with coefficients

$$\begin{aligned}(\rho_\psi)_{ijkl} &= \frac{1}{(2i)^n} \int_{\mathbb{C}^n} (\partial_i^* \partial_j^* f(z))^* \partial_k^* \partial_l^* f(z) dz^* \wedge dz \\ &= \frac{16}{(2i)^n} \int_{\mathbb{C}^n} (\psi_i - \frac{z_i}{\alpha^2})^* (\psi_j - \frac{z_j}{\alpha^2})^* (\psi_k - \frac{z_k}{\alpha^2}) (\psi_l - \frac{z_l}{\alpha^2}) |f(z)|^2 dz^* \wedge dz.\end{aligned}$$

Clearly, if $\psi_s = \psi_s^r + i\psi_s^i$, for $s = 1, \dots, n$, then

$$|f(z)|^2 = \left(\frac{2}{\sqrt{\pi\alpha}} \right)^{2n} \prod_{s=1}^n e^{-\frac{4}{\alpha^2}(x_s - \frac{\alpha^2 \psi_s^r}{2})^2} e^{-\frac{4}{\alpha^2}(y_s - \frac{\alpha^2 \psi_s^i}{2})^2}.$$

Hence, using the Fubini theorem, one can integrate real and imaginary parts separately. Using standard expressions for first four gaussian moments, one can check that

$$\begin{aligned}(\rho_\psi)_{ijkl} &= \psi_i^* \psi_j^* \psi_k \psi_l \\ &\quad + \frac{1}{\alpha^2} (\psi_i^* \psi_l \delta_{jk} + \psi_j^* \psi_k \delta_{il} + \psi_i^* \psi_k \delta_{jl} + \psi_j^* \psi_l \delta_{ik}) \\ &\quad + \frac{1}{\alpha^4} (\delta_{ik} \delta_{jl} + \delta_{jk} \delta_{il}).\end{aligned}$$

Now if we evaluate $\bar{\rho}_\psi$ at a general element $c \in \mathbb{C}^n \otimes (\mathbb{C}^n)^*$, we obtain

$$\begin{aligned}\bar{\rho}_\psi(c) &= \sum_{ijkl} (\rho_\psi)_{ijkl} c_{ij}^* c_{kl} \\ &= \left| \sum_{ij} \psi_i c_{ij} \psi_j \right|^2 + \frac{1}{\alpha^2} \sum_j \left| \sum_i \psi_i (c_{ij} + c_{ji}) \right|^2 + \frac{1}{2\alpha^4} \sum_{i,j} |c_{ij} + c_{ji}|^2.\end{aligned}$$

Therefore we see that $\bar{\rho}_\psi(c) = 0$ for every antisymmetric tensor c . On the other hand if c is not antisymmetric then the third sum of the above expression gives positive contribution to it. Thus rank of ρ_ψ equals $\frac{n(n+1)}{2}$ that is the codimension of the space of antisymmetric tensors.

8 Concluding remarks

We presented integral formulas for separable mixed states of bi-partite finite dimensional systems. The states which can be integrally represented are automatically separable. Almost all separable states (in particular, all lying in the interior of the set of such states) can be represented in the integral form.

There are natural questions related to our results.

Q1 The map Φ in the integral formula for a given state is not uniquely determined by the state. It would be advantageous to isolate a subclass of maps Φ in which the representation is unique. Does it exist such a subclass?

Q2 Can the results be generalized to $H = K \otimes L$, with infinite dimensional K or L ?

Q3 Can they be generalized to multi-partite systems?

We do not know the answer to question Q1. Answering question Q2 we see that the space $L = \mathbb{C}^n$ can not be replaced by an infinite dimensional Hilbert \tilde{L} space because there is no natural measure on \tilde{L} to be used in the integral representation. On the other hand, it is possible to replace $K = (\mathbb{C}^m)^*$ with an infinite dimensional Hilbert space \tilde{K} . In this case a map $\tilde{\Phi} : \mathbb{C}^n \rightarrow \tilde{K}^*$ should play the role of the previous map $\Phi : \mathbb{C}^n \rightarrow \mathbb{C}^m$ and statements (a) in Theorems 3.1 and 3.2 remain true, with almost the same proofs. However, statements (b) can not be true as Proposition 2.2 does not hold in the case of infinite dimension. Namely, in this case the cone of separable hermitian forms is closed, convex and nowhere dense in the space of all hermitian operators (see [15]). Statements weaker than (b) follow from our results, by taking maps $\tilde{\Phi}$ with images in finite dimensional subspaces of \tilde{K} .

Question Q3 seems to have a negative answer if we try to generalize our approach literally. However, there is a different way of representing separable states in an integral form, which works for multipartite systems, too. This approach is a subject of a forthcoming paper by the same authors.

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